QUANTIFIED RELEVANCE LOGIC AND GENERALISED RESTRICTED GENERALITY

M.W. BUNDER

After taking great pains to establish axioms for $\rightarrow$ that avoid the «paradoxes» of implication, many relevance logicians seem to accept equally paradoxical statements such as

$$(x)\,[(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))] \qquad -(1)$$

without due consideration. ((1) is an axiom of Anderson's system EQ([1]) and also of Belnap's system RQ([2]).)

If the x in the above quantifier is taken to range over a certain set S, it may be that for some x's in S some parts of (1) may be meaningless. Even if S is so chosen that every well formed formula with a free variable x is meaningful for every $x \in S$, it is clear that on some occasions either $(A \rightarrow B)$, $(B \rightarrow C)$ or A, will be false and to assert (1) on such occasion surely violates relevance.

What is required is a quantification only over the relevant x's in S, for example in (1) those for which $A \rightarrow B$ holds. This is given to us in Combinatory Logic (see [5] and [6]) where we have Ξ (restricted generality) with the rule:

Rule Ξ $\qquad \Xi XY, XU \vdash YU$.

If PWZ represents $W \rightarrow Z$ (note P could be defined in terms of Ξ as in [3]), (1) can be rewritten as:

$$\Xi[\lambda x(PAB)]\;[\lambda x(P(PBC)(PAC))],$$

or using $Wx \supset_x Zx$ to stand for ΞWZ, as

$$A \rightarrow B \supset_x (B \rightarrow C) \rightarrow (A \rightarrow C).$$

(To save on brackets we take $\supset_x$ to be a «stronger connective» than $\rightarrow$).

It could be argued that if $U \supset_x V$ is to be relevant, x should actually appear both in U and in V. In that case the combinatory logic to be used will be a λI - calculus (see [5]), i.e. one without the combinator K. (P is then not definable in terms of Ξ).

On the other hand it might be thought that as

$$A \supset A \vee x = x$$

is relevant, that

$$A \supset_x A \vee x = x$$

should be as well, even if x is not in A. In that case we use the λK - calculus of [5].

The notation that we have so far however is not sufficient to deal with multiple quantification.

$$(y)\,(x)\,[(A \to B) \to ((B \to C) \to (A \to C)], \qquad -(2)$$

for example is also an axiom of EQ and RQ but cannot be represented in terms of Ξ.

This situation can be handled with a restricted version (that for k = l) of the generalised restricted generality ${}^k\Xi^n$ introduced in [4]. This (for k = l) has the rule:

Rule ${}^l\Xi^n$ ${}^l\Xi^n\,XY,\ XU_1U_2 \ldots U_n \vdash YU_1U_2 \ldots U_n$.

If we write ${}^l\Xi^n\,XY$ as $Xu_1 \ldots u_n \supset_{u_1,\ldots,u_n} Yu_1 \ldots u_n$, (2) becomes:

$$A \to B \supset_{x,y} (B \to C) \to (A \to C).$$

A suitable rule replacing the generalisation rule of [2] which generates axioms such as (1) and (2) would then be:

If $X \supset_{u_1,\ldots,u_k} Y$ is an axiom for $k < n$ then so is

$X \supset_{u_1,\ldots,u_n} Y$ ($u_1, \ldots, u_n$ are/may be free variables in X and Y).

In this we take $X \supset_{u_1,\ldots,u_k} Y$ to be $X \to Y$ if $k = 0$.

Universal and existential quantifiers can still be defined in this system using a universal class E as in [3], so other axioms of EQ and RQ such as

$$(x)\,(A \to B) \to ((\exists x)\,A \to B)$$

and $(x)\,(A \to B) \to ((x)\,A \to (x)\,C)$

can be left in that form.

On the other hand they can also be generalised to:

$$(A \supset_x B) \rightarrow ((\exists x)\, A \rightarrow B)$$

and $(A \supset_x C) \rightarrow ((x)\, A \rightarrow (x)\, C)$

where if A is $D \rightarrow E$, $(x)A$ could be replaced by $D \supset_x E$ etc.

University of Wollongong M.W. BUNDER

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