A SEQUENCE OF NORMAL MODAL SYSTEMS WITH NON-CONTINGENCY BASES

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In [1], [2], [3], and [4], Montgomery and Routley investigate some properties of a sequence of systems T^n_Δ, $n \geqslant o$, defined by: T^n_Δ = T (specified by some suitable non-contingency basis, see [1] pp 318-319), together with the axiom $\Delta^n p$, where 'Δ^n' denotes n iterations of the non-contingency operator 'Δ' ('$\Delta^o p$' denotes 'p'). We investigate some properties of a similar sequence of systems S^n_Δ, $n \geqslant o$, defined by: S_Δ = T (specified by some non-contingency basis) together with the axiom $\Delta^n p \equiv \Delta^{n+1} p$.

Montgomery and Routley obtain the following results (where '$A \rightarrow B$' means 'The set of theorems of B is a subset (not necessarily proper) of the set of theorems of A')

1) $\sim (T^n_\Delta \rightarrow T^m_\Delta)$ for $m < n$

2) $T_\Delta \rightarrow T^m_\Delta$ for $m \geqslant n$

3) $\sim (T^n_\Delta \rightarrow S^m_\Delta)$ for $m < n$

4) $T^n_\Delta \rightarrow S^m_\Delta$ for $m \geqslant n$

We note in passing that 3) and 4) provide relative consistency results for the S^n_Δ, $n \geqslant 1$. S_Δ is of course the inconsistent system.

An *S-model* for S^1_Δ is an ordered triple $< K, R, v >$ where $K = \{H_1, \ldots, H_k\}$, k arbitrary but > 1

$R \subseteq K^2$ such that $H_i R H_j$ iff $i \leqslant j$
$v \in \{T, F\}^{W x K}$ (where W is the set of

wffs of $S^1_\triangle$) is the usual valuation function, satisfying in particular

$v(\triangle\,\alpha, H_i) = T$ iff $(j, l)\,(H_iRH_j \;\&\; H_iRH_l\,.\supset v(\alpha, H_j) = v(\alpha, H_l))$ for any $\alpha \in W$, any $H_i, H_j, H_l \in K$.

Theorem 1 Any theorem of $S^1_\triangle$ is true in all S-models.

Proof. Any S-model $<K, R, v>$ is normal, and R is reflexive, so any S-model is a model for T. It remains to be shown that $\triangle\, p \equiv \triangle\triangle\, p$ is true in all S-models. Suppose that $v(\triangle p, H_i) = T$. Then $(j, l)\,(H_iRH_j \;\&\; H_iRH_l\,.\supset v(p, H_j) = v(p, H_l))$. But H_iRH_i, so $(j)\,(H_iRH_j \supset v\,(p, H_i) = v(p, H_j))$. So for $j > i$, $v(\triangle p, H_j) = T$. So for $j \geqslant i$, $v(\triangle p, H_j) = T$. So $v(\triangle\triangle p, H_i) = T$. Thus $v(\triangle p \equiv \triangle\triangle p, H_i) = T$. In particular, since H_kRH_k, and $\sim H_kRH_l$ for $l \neq k$, $v(\triangle p, H_k) = T$, and so for $n \geqslant 1$, $v(\triangle^n, H_k) = T$ by an induction.

Suppose on the other hand that $v(\triangle p, H_i) = F$. Then for some $H_j, H_l \in K$; H_iRH_j, H_iRH_l, $v(p, H_j) = F$ and $v(p, H_l) = T$. Let $h = \min\{j, l\}$. Then since $h \leqslant j, l$, $v(\triangle p, H_h) = F$. But $v(\triangle p, H_k) = T$, and H_iRH_k. So $v(\triangle\triangle p, H_i) = F$. Thus $v(\triangle p \equiv \triangle\triangle p, H_i) = T$.

Theorem 2 $S^n_\triangle \rightarrow S^m_\triangle$, for $n \leqslant m$.

Proof. For $n \neq m$, the result follows by alternate applications of $\vdash \alpha \rightarrow\, \vdash \triangle\,\alpha$ and $\triangle\,(p \equiv q) \supset .\triangle p \equiv \triangle q$, both of which hold for T. See [1] pp 318-9.

Theorem 3 Any theorem of $S^n_\triangle$ is true in all S-models.

Proof. Follows from theorems 1 and 2.

Theorem 4. $\triangle^n p \equiv \triangle^m p$; $m, n \geqslant 1$ is true in all S-models.

Proof. Follows by an induction from theorem 3.

Theorem 5. $\triangle^n p$, $n \geqslant 1$, are false in some S-model.

Proof. An S-model which falsifies $\triangle^n p$ is as follows $K = \{H_1, H_2\}$, $R = \{<H_1, H_1>, <H_1, H_2>, <H_2, H_2>\}$, $v(p, H_1) = F$, $v(p, H_2) = T$.

Clearly $v(\triangle p, H_1) = F$ and $v(\triangle p, H_2) = T$. Whence by theorem 4, $v(\triangle^n p, H_1) = F$, for any $n \geqslant 1$.

Theorem 6. $\sim(S^n_\triangle \rightarrow T^m_\triangle)$

Proof. By theorems 3 and 5, there is a model for $S^n_\triangle$ which falsifies $\triangle^m p$.

Theorem 7. $\sim(S^n_\triangle \rightarrow S^m_\triangle)$, for $n > m$.

Proof. Follows from Montgomery and Routley's (3) and (4) above. If $S^n_\triangle \rightarrow S^m_\triangle$ for some n, m, $n > m$, then since $T^n_\triangle \rightarrow S_\triangle$, we have that $T^n_\triangle \rightarrow S^m_\triangle$ for some n, m, $n > m$, contradicting (3).

The above theorems serve to separate the $S^n_\triangle$ from one another and from the $T^m_\triangle$. The relationship between the two sequences can be diagrammed as follows:

$$\begin{array}{ccccccc} T^0_\triangle & \rightarrow & T^1_\triangle & \rightarrow & T^2_\triangle & \rightarrow & \dots \\ \updownarrow & & \downarrow & & \downarrow & & \\ S^0_\triangle & \rightarrow & S^1_\triangle & \rightarrow & S^2_\triangle & \rightarrow & \dots \end{array}$$

Theorem 8. $S^n_\triangle$, for $n \geqslant 1$, has $2(n + 1)$ modalities.

Proof. Theorem 2 above shows that $S^n_\triangle$ has at most $2(n + 1)$ We need to show that $\triangle^i p \equiv {\sim}\triangle^j p$, for $i, j \leqslant n$, are not provable in $S^n_\triangle$. Suppose the contrary and let $i = \min\{i, j\}$. Then since $S^i_\triangle \rightarrow S^n_\triangle$, $\triangle^i p \equiv {\sim}\triangle^j p$ is provable in $S^i_\triangle$. But $\triangle^i p \equiv \triangle^j p$ is provable in $S^i_\triangle$, making $S^i_\triangle$, and $T^i_\triangle$, inconsistent.

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