PREDICATE CALCULUS WITHOUT FREE VARIABLES

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In this paper we develop a predicate calculus in which all well formed formulas are closed and which therefore requires no generalisation rule. We show that the closure of any standard predicate calculus theorem is provable in this system.

In this closed system we cannot obtain predicate calculus formulas by straight substitution into propositional calculus theorems. The axioms of propositional calculus, therefore, have to be changed as follows:

1. $\vdash (x_1)\ldots(x_n): A\supset.B\supset A,$

where $x_1, \ldots, x_n$ include the free variables in A and B.

2. $\vdash (x_1)\ldots(x_n): A\supset.B\supset C:\supset:A\supset B.\supset.A\supset C,$

where $x_1, \ldots, x_n$ include the free variables in A, B and C.

3. $\vdash (x_1)\ldots(x_n): \sim A\supset \sim B.\supset.B\supset A,$

where $x_1, \ldots, x_n$ include the free variables in A and B.

A, B, C, etc in the above stand for well formed formulas in the ordinary predicate calculus; here we could call them *well formed predicate expressions* (wfpes).

In the case where the wfpes have no free variables the above axioms can reduce to the standard propositional calculus axioms.

Also we require a predicate calculus version of modus ponens ("*closed modus ponens*"):

If $\vdash (x_1)\ldots(x_n).A\supset B$, where $x_1, \ldots, x_n$ include the free variables in $A\supset B$, and $\vdash (x_i)\ldots(x_j)A$, where $x_i, \ldots, x_j$ include the free variables in A, then $\vdash (x_k)\ldots(x_l)B$, where $x_k, \ldots, x_l$ include the free variables of B.

Note that this version of modus ponens, together with

Axioms 1 and 2 allows us to prove $\vdash (x_p)\ldots(x_q)A \supset A$, where $x_p, \ldots, x_q$ include the free variables of A.

If then we have $\vdash (x_i)\ldots(x_j)A$, where $x_i, \ldots, x_j$ also include the free variables of A, we can by this modus ponens, conclude $\vdash (x_r)\ldots(x_s)A$, where $x_r, \ldots, x_s$ include the free variables of A.

If $x_r, \ldots, x_s$ are part of $x_i, \ldots, x_j$, then we have dropped some vacuous quantifiers. If $x_i, \ldots, x_j$ are part of $x_r, \ldots, x_s$, we have generalised with respect to some variable; this quantification is of course also vacuous.

Alternatively we could state the axioms and modus ponens with the variables ranging only over the free variables of the wfpes concerned. In this case Theorem 1 below can only be proved for formulas containing no vacuous quantifiers. The rest of this work can easily be altered to satisfy this alternative.

Now we state the axioms proper to the predicate calculus. There is one more than the usual two, this allows us to prove theorems otherwise proved using the generalisation rule.

4. $\vdash (x_1)\ldots(x_n).(x)A \supset A^*$,

where $x_1, \ldots, x_n, x$ include the free variables in A, and A^* is the result of replacing all free occurrences of x in A by a term having some or all of $x_1, \ldots, x_n$ as free variables, x in A however must not be in the scope of one of the quantifiers $(x_1), \ldots, (x_n)$.

5. $\vdash (x_1)\ldots(x_n):(x).A \supset B: \supset :A \supset (x)B$,

where $x_1, \ldots, x_n, x$ include the free variables of $A \supset B$, x being not free in A.

6. $\vdash (x_1)\ldots(x_n).(x)(y)A \supset (y)(x)A$,

where $x_1, \ldots, x_n, x, y$ include the free variables of A.

We call the predicate calculus based on these axioms and closed modus ponens, "*closed predicate calculus*". We now prove:

Theorem 1: If $\vdash D$ is a theorem of first order predicate calculus, where $y_1, \ldots, y_n$ are the free variables of D, then

$$\vdash (y_1)\ldots(y_m)D$$

is a theorem of closed predicate calculus.

Proof: Consider ordinary first order predicate calculus based on the propositional calculus axioms similar to Axioms 1, 2 and 3, modus ponens, generalisation,

(a) $\vdash (x_i)\ A(x_i) \supset A(t)$,

where $A(x_i)$ is a well formed formula (wf) and t is a term of the system free for x_i in $A(x_i)$, and

(b) $\vdash (x_i)(A \supset B) \supset (A \supset (x_i)B)$,

where A is a wf of the system, containing no free occurrences of x_i.

We then perform the proof of the theorem by induction on the proof of $\vdash D$. Consider first the steps of the proof which are instances of the axioms of first order predicate calculus.

If the axiom is a propositional calculus axiom, then any instance of it is given by Axiom 1, 2 or 3.

If the axiom is (a), the closed version is our Axiom 4. If it is (b), we have Axiom 5.

In the inductive step assume that all the steps in the proof to a certain point have corresponding statements provable in the closed predicate calculus.

If the next step is made using two earlier statements and modus ponens, i.e. $A \supset B$ and A then B, then by our inductive hypothesis we have as theorems of the closed predicate calculus:

$\vdash (x_1)\ldots(x_n).A \supset B$,

where $x_1, \ldots, x_n$ are the free variables of $A \supset B$ and

$\vdash (x_k)\ldots(x_1).A$,

where $x_k, \ldots, x_1$ are the free variables of A.

Then by closed modus ponens we have:

$\vdash (x_p) \ldots (x_q)B,$

where $x_p, \ldots, x_q$ are the free variables in B.

This is the closure of $\vdash B$.

If the next step is made by generalising $\vdash A$ to $\vdash (x)A$, then by the inductive step we have:

$\vdash (x_1) \ldots (x_n)A,$

where $x_1, \ldots, x_n$ are the free variables in A.

If x is one of these, say x_k, then by repeated use of Axiom 6 and closed modus ponens we obtain:

$\vdash (x_1) \ldots (x_{k-1})(x_{k+1}) \ldots (x_n)(x)A,$

which is what is required.

If x is not one of $x_1, \ldots, x_n$, we can of course generalise using closed modus ponens and shift x to the front as before.

Thus the theorem is proved in all cases.

Closed predicate calculus can now be applied whenever ordinary first order predicate calculus could be applied. Where, such as in the proof of Gödels Theorem (see [1]), we use a well formed formula containing a free variable, we must of course use a well formed predicate expression.

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BIBLIOGRAPHY

[1] Mendelson — *Introduction to Mathematical Logic.* New York 1964.