MODALITIES IN A SEQUENCE OF NORMAL NON-CONTINGENCY MODAL SYSTEMS

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Some properties of sequences of systems lying between T and S4, and T and S5, having added axioms of the form $\Box^n p \supset \Box^{n+1} p$ have been investigated in [4] and elsewhere (¹). These systems have infinitely many modalities for $n > 1$. Here we consider an analogous sequence of noncontingency based systems, the i-th member of the sequence being denoted by $T_\Delta{}^i$ where i is any positive integer. The primitive basis of $T_\Delta{}^i$ is given by adding the axiom $\Delta^i p$ (where 'Δ^i' denotes i iterations of the noncontingency modal connective 'Δ', and '$\Delta^\circ p$' denotes 'p') to either $S1_\Delta$ (see [3]) or to one of the noncontingency based formulations of T given in [2]. The equivalence of these bases follows from Theorem 5. of [3]. The first member of this sequence, $T_\Delta{}^1$, is deductively equivalent to the Trivial System, the second, $T_\Delta{}^2$, to S5, and the remaining members of the infinite sequence lie between S5 and T. Each system $T_\Delta{}^i$ has $2(i + 1)$ distinct modalities.

A T_Δ-model of degree n is an ordered triple $<K_n, R_n, V_n>$ where:

(i) $K_n = \{H_1 \ldots . H_n\}$

(ii) R_n is a relation over K_n such that $H_i R_n H_j$ holds iff $j \geqslant i - 1$.

(iii) V_n is a valuation function from wffs of T_Δ and members of K_n on to $\{t, f\}$, defined as follows:

If P is any variable, A, B any wffs of T_Δ, and H_i, H_j, H_k members of K_n then:

(a) For each P and each H_i either $V_n(P,H_i) = t$ or $V_n(P,H_i) = f$.

(b) $V_n(\sim P,H_i) = t$ iff $V_n(A,H_i) = t$.

(c) $V_n(A \supset B,H_i) = f$ iff $V_n(A,H_i) = t$ and $V_n(B,H_i) = f$.

(¹) See remarks and footnotes in [1], pp. 259-260.

(d) $V_n(\Delta A, H_i) = f$ iff for some H_j and $H_k : H_i R_n H_j$, $H_i R_n H_k$, $V_n(A, H_j) = t$ and $V_n(A, H_k) = f$.

A wff A is true in a T_Δ-model iff for every H_i in the K of that model, $V(A, H_i) = t$. Otherwise A is false in that model.

Theorem 1. Every theorem A of $T_\Delta{}^n$ is true in every T_Δ-model of degree n where $n \geqslant 1$.

Proof: Since R_n is reflexive, every T_Δ-model of any degree is a T-model in the usual sense. Thus it remains to show that $\Delta^n p$ is true in every T_Δ-model of degree n where $n \geqslant 1$.

We consider the two possible cases:

Either (i) for all H_i in K_n, $V_n(\Delta p, H_i) = t$.

or (ii) for some H_i in K_n, $V_n(\Delta p, H_i) = f$.

Case (i) The result is immediate by $n - 1$ applications of clause (d) of the definition of V_n.

Case (ii) Let 1 be the largest value of i for which $V_n(\Delta p, H_i) = f$. Then by the definition of the model and the hypothesis, $V_n(\Delta p, H_j) = f$ iff $1 \leqslant j \leqslant 1$. Also, by $k - 1$ applications of clause (d) of the definition, for $1 \leqslant k \leqslant n - 1$, $V_n(\Delta^k p, H_j) = f$ iff $1 \leqslant j \leqslant 1 + k - 1$.

Hence, for $n = 1 + k - 1$

$$V_n(\Delta^{n-1+1} p, H_j) = f \text{ iff } 1 \leqslant j \leqslant n.$$

Hence, by the definition

$$V_n(\Delta^{n-1+2} p, H_j) = t \text{ iff } 1 \leqslant j \leqslant n,$$

and by $1 - 2$ applications of clause (d),

$$V_n(\Delta^n p, H_j) = t \text{ iff } 1 \leqslant j \leqslant n.$$

Note that by the hypothesis of the case and the definition of the model, $1 \geqslant 2$.

Theorem 2. $\Delta^{n-1} p$ is false in some T_Δ-model of degree n.

Proof: Consider the following model: $V_n(p, H_i) = f$ iff $i = 1$. Then if $n = 1$, $\Delta^n p$ is false in this model. If $n > 1$, then proceeding as in case (ii) of the proof of theorem 1, $1 = 2$ by the definition of V_n, and so

$$V_n(\Delta^{n-1} p, H_j) = f \text{ iff } 1 \leqslant j \leqslant n.$$

Hence $V_n(\Delta^{n-1}p,H_j) = f$ for every H_j in K_n, and $\Delta^{n-1}p$ is false in this model.

Theorem 3. If $O \leqslant i < j \leqslant n$ then $\Delta^i p \equiv \Delta^j p$ is false in some T_Δ-model of degree n.

Proof: Consider the model where $V_n(p,H_m) = f$ iff $1 \leqslant m \leqslant n - j + 1$.

Then $V_n(\Delta^j p,H_k) = t$ for every H_k in K_n, and for $i<j$, $V_n(\Delta^i p,H_k) = f$ for some H_k in K_n. Hence for some H_k in K_n, $V_n(\Delta^j p \supset \Delta^i p,H_k) = f$, and so $\Delta^i p \equiv \Delta^j p$ is false in some T_Δ-model of degree n.

Theorem 4. The system $T_\Delta{}^n$ has $2(n + 1)$ irreducible modalities.

Proof: It is easily shown by the matrix:

$\supset$	1	2	$\sim$	Δ
*1	1	2	2	1
2	1	1	1	1

that no wff of the form $\Delta^i p \equiv \sim \Delta^j p$ is provable in $T_\Delta{}^n$. It follows from theorems 2. and 3. above that the system $T_\Delta{}^n$ has no pair of modalities from; p, $\sim p$, Δp, $\sim \Delta p, \ldots \Delta^n p$, $\sim \Delta^n p$ equivalent, and by theorem 6. of [3] these are all the irreducible modalities in the system.

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