NON-CONTINGENCY AXIOMS FOR S4 AND S5

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In [3] some contingency and non-contingency bases were developed for normal modal logics. Some of the axioms presented there for extending **T** to **S4** and **S5** are sufficiently strong to give these latter systems when added to the non-normal system **S3**.

The notation of [3] is adopted except that 'Feys' is here further abbreviated to 'F'. 'SD' refers to the rule of detachment for strict implication and '$\rightarrow$' symbolises strict implication.

It is well-known that Gödel's A3, $\Box p \supset \Box\Box p$, added to **S3** gives a system deductively equivalent to **S4**, but that A4, $\Diamond p \supset \Box\Diamond p$ when added to **S3** gives a system weaker than **S5**, namely **S3.5** (Aqvist [1]). However, if instead of A4 the axiom

A5. $\Diamond\Box p \supset \Box\Box p$

is used it is easily seen that both A3 and A4 are deducible, and that A5 is provable in **S5**. For by F36.0, F37.2, A5, SD and SL we have A3; by F37.12, F37.2, A5, SD and SL we have $\Diamond\Box p \supset \Box p$, a contraposed form of A4; and from this and A3, by SL we have immediately A5. Hence (**S3**+A5) is deductively equivalent to **S5**.

By SL an equivalent form of A5 is

A6. $\Box\Box p \vee \Box \sim \Box p$

In terms of the non-contingency operator 'Δ' defined by

$$\Delta A =_{df} \Box A \vee \Box \sim A$$

the axiom A6 becomes

S52. $\Delta\Box p$

So if the system **S3** be thought of as having the above Df Δ

added, the system (**S3** + S52) is seen to be deductively equivalent to **S5**.

Consider the following axioms:

S41. $\Delta p \supset \Delta \Delta p$
S44. $\Delta p \supset \Delta \Box p$
S51. $\Delta \Delta p$
S52. $\Delta \Box p$

Theorem 1. (**S3** + S41) is deductively equivalent to **S4**.
Proof: It suffices to derive A3 in (**S3** + S41), the theorem then follows from standard results and theorem 8 of [3].

1.	$\Delta p \supset \Delta \Delta p$	S41
2.	$\Box p \supset \Delta \Delta p$	1, DfΔ, SL.
3.	$\Box p \supset \Box \Delta p \vee \Box \sim \Delta p$	2, DfΔ
4.	$\Box p \vee \Box \sim p \supset \Delta p$	SL, DfΔ.
5.	$\Box p \supset \Delta p$	4, SL.
6.	$\Delta p \supset \sim \Box \sim \Delta p$	F37. 12, F36.0, SD.
7.	$\Box p \supset \sim \Box \sim \Delta p$	5, 6, SL.
8.	$\Box p \supset \Box \Delta p$	3, 7, SL.
9.	$\Delta p \rightarrow \Box p \vee \Box \sim p$	F31.11, DfΔ.
10.	$\Delta p \rightarrow \sim \sim \Box \sim p \vee \Box p$	9, SL, F34.42, SE.
11.	$\Delta p \rightarrow . \Diamond p \supset \Box p$	10, SL, F34.42, SE.
12.	$\Box \Delta p \supset . \Diamond p \rightarrow \Box p$	F33.311, 11, SD.
13.	$\Diamond p \rightarrow \Box p \rightarrow . \Box \Diamond p \supset \Box \Box p$	F33.311.
14.	$\Diamond p \rightarrow \Box p \supset . \Box \Diamond p \supset \Box \Box p$	F37.12, F32.02, 13, SD.
15.	$\Box p \supset . \Box \Diamond p \supset \Box \Box p$	8, 12, 14, SL.
16.	$\Box p \supset \Box \Diamond p$	F33.311, F36.0, SD.
17.	$\Box p \supset \Box \Box p$	15, 16, SL.

Theorem 2. (**S3** + S51) is deductively equivalent to **S5**.
Proof: Since $\Delta \Delta p \supset . \Delta p \supset \Delta \Delta p$ is a theorem of **S3** by SL, it follows by theorem 1 that (**S3** + S51) deductively includes (**S3** + S41) and hence also system **T**. It follows by theorem 13 of [3] that (**S3** + S51) is deductively equivalent to **S5**.

Theorem 3. (**S3** + S44) is deductively equivalent to **S4**.
Proof: It suffices to show that S44 is a theorem of **S4** and that A3 is a Theorem of (**S3** + S44).

ad S44
1. $\Box p \supset \Box\Box p$ A3.
2. $\sim p \to \sim \Box p$ F31.34, F37.12, SD.
3. $\Box \sim p \supset \Box \sim \Box p$ F33.311, 2, SD.
4. $\Box p \vee \Box \sim p \supset \Box\Box p \vee \Box \sim \Box p$ 1, 3, SL.
5. $\Delta p \supset \Delta \Box p$ 4, Df Δ.

ad A3.
1. $\Delta p \supset \Delta \Box p$ S44.
2. $\Box p \vee \Box \sim p \supset \Box\Box p \vee \Box \sim \Box p$ 1, Df Δ.
3. $\Box p \supset \Box\Box p \vee \Box \sim \Box p$ 2, SL.
4. $p \supset \sim \Box \sim p$ F36.0, F34.2.
5. $\Box p \supset \sim \Box \sim \Box p$ 4.
6. $\Box p \supset \Box\Box p$ 3, 5, SL.

Theorem 4. (**S3** + S52) is deductively equivalent to **S5**.
Proof: A proof has already been sketched in the introductory remarks above. It also follows from Theorem 3 above and the results in [3].

It is not known whether the **S3** base can be further weakened. A3 is provable in ($\mathbf{S1}^{\circ} + p \supset \Diamond p$ + S44), but neither $p \to \Diamond p$ nor $\Box p \to \Box\Box p$ appear to be provable in this system; **T** is deductively included in (**S1** + A3) but this system may be weaker than **S4**; **S3** appears not to be included in (**S1** + A4).

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