AN ALTERNATIVE CHARACTERISATION OF FIRST-DEGREE ENTAILMENT

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Let α and β be formulae constructed from proposition letters p, q, r, ... by means of the operators N, K, and A (for negation, conjunction, and disjunction). Classically, we say that α *tautologically implies* β, and write $\alpha \vdash \beta$, if there is no assignment h of truth-values to proposition letters such that $h(\alpha) = T$ whilst $h(\beta) = F$. As is well known, many philosophers and logicians have felt that tautological implication fails to capture certain intuitive concepts of logical involvement or 'entailment' between truth-functional formulae, and have offered other accounts of such concepts.

One particularly elegant and interesting kind of entailment relation has recently been studied by A. R. Anderson and N. D. Belnap (1). They define *first-degree entailment* between formulae in N, K, A, to be the least relation $\rightarrow$ which satisfies the following conditions:

$K\alpha\beta \rightarrow \alpha$ $\qquad\qquad$ $\alpha \rightarrow A\alpha\beta$

$K\alpha\beta \rightarrow \beta$ $\qquad\qquad$ $\beta \rightarrow A\alpha\beta$

$\alpha \rightarrow NN\alpha$

$NN\alpha \rightarrow \alpha$

$K\alpha A\beta\gamma \rightarrow AK\alpha\beta\gamma$

If $\alpha \rightarrow \beta$ and $\beta \rightarrow \gamma$ then $\alpha \rightarrow \gamma$

If $\alpha \rightarrow \beta_1$ and $\alpha \rightarrow \beta_2$ then $\alpha \rightarrow A\beta_1\beta_2$

If $\alpha_1 \rightarrow \beta$ and $\alpha_2 \rightarrow \beta$ then $A\alpha_1\alpha_2 \rightarrow \beta$

If $\alpha \rightarrow \beta$ then $N\beta \rightarrow N\alpha$

In this note we give an alternative characterisation of first-degree entailment, via the concept of a 'simple formula'.

We say that a formula is *simple* if (2) no proposition letter occurs more than once in it.

(1) See the references in the bibliography, especially [4] pp. 92-96 and [1]. The terminology of Anderson and Belnap is not entirely uniform: we take the name «first-degree entailment» from [1], but in [4] the same relation is called «basic implication».

We say that a formula α *simply implies* a formula β if there are simple formulae α' and β', with the pair (α, β) a substitution instance of the pair (α', β'), such that $\alpha' \vdash \beta'$.

Simple implication does not itself coincide with first-degree entailment; it is a very weak relation with relatively few closure properties. It is not the case that for all formulae α, β_1, β_2, if α simply implies each of β_1 and β_2 then α simply implies $K\beta_1\beta_2$. For example whilst we have:

p simply implies p

and so, to repeat ourselves,

p simply implies p,

we do not have:

p simply implies Kpp.

If however we close simple implication under transitivity, consequent-conjunction, and antecedent-disjunction, we have an entirely different picture.

Let us define *structural implication* to be the least relation $\Rightarrow$ between formulas in N, K, A, such that:

If α simply implies β then $\alpha \Rightarrow \beta$

If $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \gamma$ then $\alpha \Rightarrow \gamma$

If $\alpha \Rightarrow \beta_1$ and $\alpha \Rightarrow \beta_2$ then $\alpha \Rightarrow K\beta_1\beta_2$

If $\alpha_1 \Rightarrow \beta$ and $\alpha_2 \Rightarrow \beta$ then $A\alpha_1\alpha_2 \Rightarrow \beta$

From the intuitive point of view, structural implication is thus the least relation which includes simple implication and which allows certain manipulations which are necessary for its use in any kind of sequential deduction.

We shall show that structural implication coincides with first-degree entailment.

Lemma 1

For all formulae α and β, if $\alpha \rightarrow \beta$ then $\alpha \Rightarrow \beta$

Proof

It is clear that for each direct clause in the definition of $\rightarrow$, the left-hand formula simply implies, and so structurally implies, the right-hand formulae. Moreover, each closure condition in the definition of $\rightarrow$, execept the last, is also a closure condition in the definition of $\Rightarrow$. Hence to prove the lemma it suffices to show that for all formulae α and β, if $\alpha \Rightarrow \beta$ then $N\beta \Rightarrow N\alpha$, for which we induce on the definition of $\Rightarrow$.

(2) *i.e.*: if and only if.

For the basis, suppose that α simply implies β. Then there are simple formulae α' and β', with (α, β) a substitution instance of (α', β'), such that $\alpha' \vdash \beta'$. But then $N\beta'$, and $N\alpha'$ are simple formulae, with the pair $(N\beta, N\alpha)$ a substitution instance of $(N\beta', N\alpha')$, such that $N\beta' \vdash N\alpha'$. Hence $N\beta$ simply implies $N\alpha$, and so $N\beta \Rightarrow N\alpha$.

In the induction step we have three cases to consider.

For the first case suppose that $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \gamma$; we need to show that $N\gamma \Rightarrow N\alpha$. By the induction hypothesis, since $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \gamma$ we have $N\beta \Rightarrow N\alpha$ and $N\gamma \Rightarrow N\beta$. Hence by the transitivity of $\Rightarrow$, $N\gamma \Rightarrow N\alpha$.

For the second case of the induction step, suppose that $\alpha \Rightarrow \beta_1$ and $\alpha \Rightarrow \beta_2$; we need to show that $NK\beta_1\beta_2 \Rightarrow N\alpha$. Since $\alpha \Rightarrow \beta_1$ and $\alpha \Rightarrow \beta_2$ we have by the induction hypothesis that $N\beta_1 \Rightarrow N\alpha$ and $N\beta_2 \Rightarrow N\alpha$. Hence by a closure condition in the definition of $\Rightarrow$, $AN\beta_1N\beta_2 \Rightarrow N\alpha$. But clearly $NK\beta_1\beta_2$ simply implies $AN\beta_1N\beta_2$, and so $NK\beta_1\beta_2 \Rightarrow AN\beta_1N\beta_2$. Hence by the transitivity of $\Rightarrow$, $NK\beta_1\beta_2 \Rightarrow N\alpha$.

A similar argument suffices to establish the third case of the induction step.

Lemma 2

For all formulae α and β, if $\alpha \Rightarrow \beta$ then $\alpha \rightarrow \beta$.

Proof

Since all of the closure conditions used in the definition of $\Rightarrow$ already occur as closure conditions in the definition of $\rightarrow$, it suffices to show that if α simply implies β then $\alpha \rightarrow \beta$.

Suppose then that α simply implies β. Then there are simple formulae α' and β', with (α, β) a substitution instance of (α', β'), such that $\alpha' \vdash \beta'$.

Let $m(\alpha')$ and $m(\beta')$ be the formulae obtained from α' and β' by applying de Morgan transformations and double negation eliminations until no occurrence of the negation operator has more than a single proposition letter as its scope. Then by well known properties of tautological implication, since $\alpha' \vdash \beta'$ we have $m(\alpha') \vdash m(\beta')$. Also it is trivial to verify by induction on the number of de Morgan transformations and double negation eliminations performed, that since α' and β' are simple formulae, $m(\alpha')$ and $m(\beta')$ are also simple formulae.

Now let $d(m(\alpha'))$ be the formula obtained from $m(\alpha')$ by distributing K over A until we reach a formula $A\ldots A\gamma_1\ldots\gamma_x$ in disjunctive normal form. Similarly let $c(m(\beta'))$ be the formula obtained from $m(\beta')$ by distributing A over K until we reach a formula $K\ldots K\delta_1\ldots\delta_y$ in conjunctive normal form. Then each of $\gamma_1, \ldots, \gamma_x$ is itself a conjunction of letters and negations of letters, and each of $\delta_1, \ldots \delta_y$ is a disjunction of letters and negations of letters. Clearly since $m(\alpha') \vdash m(\beta')$ we have $d(m(\alpha')) \vdash c(m(\beta'))$, and so $\gamma_i \vdash \delta_j$, for all $i \leqslant x$, $j \leqslant y$.

But also it is trivial to verify by induction on the number of distributions performed in the construction of $d(m(\alpha'))$ and $c(m(\beta'))$ that since $m(\alpha')$ and $m(\beta')$ are both simple, each γ_i and likewise each δ_j is simple. Since each γ_i and each δ_j is simple, no letter occurs both negated and unnegated in any γ_i, and no letter occurs both negated and unnegated in any δ_j. Hence for each pair γ_i and δ_j, since $\gamma_i \vdash \delta_j$, there must be some letter which occurs unnegated in both of γ_i and δ_j, or else occurs negated in both of γ_i and δ_j. But then, by a well known property of first-degree entailment (3), we have $\gamma_i \rightarrow \delta_j$ for all $i \leqslant x$ and $j \leqslant y$. Hence by the closure conditions of the definition of first-degree entailment, $A \ldots A\gamma_1 \ldots \gamma_x \rightarrow K \ldots K\delta_i \ldots \delta_y$. That is, $d(m(\alpha')) \rightarrow c(m(\beta'))$. Hence by further known properties of first-degree entailment (3), $m(\alpha') \rightarrow m(\beta')$, and so $\alpha' \rightarrow \beta'$, and so $\alpha \rightarrow \beta$. This completes the proof of the lemma.

From lemmas 1 and 2 we have immediately the desired:

Theorem

For all formulas α and β, $\alpha \rightarrow \beta$ if $\alpha \Rightarrow \beta$.

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(3) See [1].