A NOTE ON THE INTUITIONIST AND THE CLASSICAL PROPOSITIONAL CALCULUS

EVERT W. BETH AND HUGUES LEBLANC

Consider the following version (PC) of the propositional calculus:

(a) The primitive signs of PC are to be: a denumerably infinite list of propositional variables, the two connectives «$\sim$» and «$\supset$», and the two parentheses «(» and «)»;

(b) The formulas of PC, referred to hereafter by «A», «B», «C», and «D», are to be all finite sequences of primitive signs of PC;

(c) The well-formed formulas (wffs) of PC are to be: all propositional variables, all formulas of the form $\sim A$, where A is a wff of PC, and all formulas of the form $(A \supset B)$, where A and B are wffs of PC.

It is clear that all the wffs of PC which are intuitionistically valid can be obtained by means of the following rules, the first four of which are structural rules, the last four elimination and introduction rules:

R1. $A \vdash A$ (Reflexivity);

R2. If $A_1, A_2, \ldots, A_n \vdash B$, then $A_1, A_2, \ldots, A_{n+1} \vdash B$ (Expansion);

R3. If $A_1, A_2, \ldots, A_n \vdash B$, then $A_i, A_2, \ldots, A_{i-1}, A_1, A_{i+1}, \ldots, A_n \vdash B$ (Permutation);

R4. If $A_1, A_2, \ldots, A_n \vdash A_{n+1}$ and $A_1, A_2, \ldots, A_{n+1} \vdash B$, then $A_1, A_2, \ldots, A_n \vdash B$ (Transitivity);

R5. $A, \sim A \vdash B$ (Weak elimination rule for «$\sim$»);

R6. If $A_1, A_2, \ldots, A_{n+1} \vdash B$ and $A_1, A_2, \ldots, A_{n+1} \vdash \sim B$, then $A_1, A_2, \ldots, A_n \vdash \sim A_{n+1}$ (Introduction rule for «$\sim$»);

R7. $A, A \supset B \vdash B$ (Weak elimination rule for «$\supset$»);

R8. If $A_1, A_2, \ldots, A_{n+1} \vdash B$, then $A_1, A_2, \ldots, A_n \vdash A_{n+1} \supset B$ (Introduction rule for «$\supset$») [1].

[1] Throughout the above rules n may of course be equal to 0 as well as larger than 0.

It is clear also that all the wffs of PC which are classically valid can be obtained by means of R1-R4, R6-R8, and the following elimination rule for « $\sim$ »:

R5′. $\sim \sim A \vdash A$ (Strong elimination rule for « $\sim$ »).

We wish to prove here that the self-same wffs, that is, the wffs of PC which are classically valid, can be obtained by means of R1-R6, R8, and the following elimination rule for « $\supset$ »:

R7′. $A \supset B, (A \supset C) \supset A \vdash B$ (Strong elimination rule for « $\supset$ »)([2]).

We first prove that R7 follows from R1-R6, R7′, and R8.

R7. $A, A \supset B \vdash B$.

(1)	$A, A \supset B, A \supset C \vdash A$	(R1 and R2),
(2)	$A, A \supset B \vdash (A \supset C) \supset A$	(R8 and (1)),
(3)	$A, A \supset B, (A \supset C) \supset A \vdash B$	(R7′, R2, and R3),
(4)	$A, A \supset B \vdash B$	(R4, (2), and (3)) ([3]).

We next prove that R5′ follows from R1-R6, R7′, and R8.

R5′. $\sim \sim A \vdash A$.

(1)	$A \supset \sim A, A \vdash A$	(R1, R2, and R3),
(2)	$A \supset \sim A, A \vdash \sim A$	(R7 and R3) ([4]),
(3)	$A \supset \sim A \vdash \sim A$	(R6, (1), and (2)),
(4)	$\sim \sim A, A \supset \sim A \vdash \sim A$	(R2, R3, and (3)),
(5)	$\sim \sim A, A \supset \sim A \vdash \sim \sim A$	(R1 and R2),
(6)	$\sim \sim A \vdash \sim (A \supset \sim A)$	(R6, (4), and (5)),
(7)	$\sim \sim A, A \supset \sim A \vdash \sim (A \supset \sim A)$	(R2 and (6)),
(8)	$\sim \sim A, A \supset \sim A, \sim (A \supset \sim A) \vdash A$	(R5, R2, and R3),
(9)	$\sim \sim A, A \supset \sim A \vdash A$	(R4, (7), and (8)),

([2]) R7′ was suggested to one of the writers by Professor Stig Kanger. For additional data on R7′, see Leblanc, H., *On chances and estimated chances of being true,* in *Revue Philosophique de Louvain,* vol. 57, May 1959, pp. 228-229.

([3]) The above proof of R7 was again suggested to one of the writers by Professor Stig Kanger.

(10)	$\sim\sim A \vdash (A \supset \sim A) \supset A$	(R8 and (9)),
(11)	$\sim\sim A, (A \supset \sim A) \supset A, A \supset A \vdash A$	(R7′, R2, and R3),
(12)	$\sim\sim A, (A \supset \sim A) \supset A \vdash A \supset A$	(R1, R8, R2, and R3),
(13)	$\sim\sim A, (A \supset \sim A) \supset A \vdash A$	(R4, (11), and (12)),
(14)	$\sim\sim A \vdash A$	(R4, (10), and (13)).

The result we have just obtained may throw additional light on the relationship between the classical propositional calculus and the intuitionist one. The classical propositional calculus has frequently been described as an intuitionist propositional calculus with a strengthened elimination rule for « ~ ». In view of our result the classical propositional calculus may likewise be described as an intuitionist propositional calculus with a strengthened elimination rule for « $\supset$ ».

University of Amsterdam — Evert W. BETH
Bryn Mawr College — and Hughes LEBLANC

(4) Our use of R7 here is legitimate, since R7 has already been shown to follow from R1-R6, R7′, and R8.